AUTOMORPHISMS IN c_0, l_1 AND m

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ABSTRACT

It is proved that if $X = c_0$ or m and if Y and Z are subspaces of X with X/Y and X/Z non-reflexive, then any isomorphism of Y onto Z has an extension to an automorphism of X. A dual result is obtained for $X = l_1$.

1. **Introduction.** The general setting of the problem we intend to study in the present paper is the following: Let X be a Banach space and let Y be a closed subspace of X. Let T be a linear isomorphism (i.e. a bounded one-to-one operator with closed range) from Y into X. Under what conditions can T be extended to an automorphism (i.e. surjective isomorphism) of X. The answer to this question is simple if the dimension of either X/Y or X/TY is finite. In this case it is obvious that a necessary and sufficient condition for the existence of $\tilde{T}$ is that $\dim X/TY = \dim X/Y$. We restrict ourselves therefore to the study of the question posed above under the assumption that X/Y and X/TY are both infinite-dimensional.

Since every subspace of the Hilbert space l_2 is a complemented subspace and every infinite dimensional subspace of l_2 is isomorphic to l_2 it is obvious that if X is l_2 the required automorphism $\tilde{T}$ always exists under the assumption we made. Our first result is that the space c_0, of all sequences of scalars tending to 0 with the sup norm, also has this property. The proof of this fact, as well as the proof of the similar results described below for l_1 and m, uses most of the known special properties of those spaces. We conjecture that this result gives a joint characterization of c_0 and l_2 among all Banach spaces and that actually the following stronger assertion can be made: The spaces c_0 and l_2 are the only "subspace homogeneous" Banach spaces. We call a Banach space X subspace homogeneous if for every pair of isomorphic subspaces Y and Z of X of infinite co-

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dimension, there is an automorphism τ of X such that $\tau(Y) = Z$. If this conjecture is verified it would certainly have significant consequences in the general theory of Banach spaces. At present, however, the interest in the result concerning c_0 (and the similar ones concerning l_1 and m) may emerge from the fact that in several arguments in linear and even non-linear functional analysis which could till now be carried through only for l_2 may now be generalized to one of the three spaces c_0, l_1 or m.

The result we prove for $l_1 = c_0^*$ is the dual to that we prove for c_0. In particular l_1 has the property which we call "quotient homogeneous" which is the natural dual of the property subspace homogeneous. Again we conjecture that only l_1 and l_2 are quotient homogeneous. The third and least elementary of the results of Section 2 gives the answer to the question we stated above for $X = m = l_1^*$. The situation here is more involved; the question of reflexivity of X/Y and X/TY and an integer valued invariant of T (which generalizes the invariant $\dim X/TY - \dim X/Y$) enter into the solution. Section 2 ends with a theorem which gives a necessary condition on a subspace Y of m to be such that m/Y is reflexive.

The study of the question treated here was initiated by a problem posed by Berg [2] (cf. Remark 3 following Theorem 3).

Section 3 is devoted to some remarks and a discussion of open problems (including the conjectures we already mentioned above).

Notations. We follow the standard notations (cf. [3] and [4]). Let us only mention that $X \approx Y$ means that X is isomorphic to Y. A subspace Y of a Banach space X is said to be complemented in X if there is a projection from X onto Y. By operator (resp. projection) we shall always understand a bounded linear operator (resp. projection). The restriction of an operator T defined on X to a subspace Y of X is denoted by $T_{|Y}$.

2. The main results.

We state first our result concerning c_0.

THEOREM 1. *Let Y be a closed subspace of c_0 and let T be an isomorphism of Y into c_0, such that $\dim c_0/Y = \dim c_0/TY = \infty$. Then there is an automorphism $\tilde{T}$ of c_0 with $\tilde{T}_{|Y} = T$.*

For the proof we need several known facts concerning c_0.

LEMMA 1. a) *Let $Z \supset Y$ be separable Banach spaces and let T be an operator from Y into c_0. Then T has an extension T_0 to an operator from Z to c_0.*

b) *An infinite dimensional subspace of* c_0 *is complemented in* c_0 *if and only if it is isomorphic to* c_0.

c) *Let* Y *be a closed subspace of* c_0 *with* $\dim c_0/Y = \infty$. *Then there is a projection* P *in* c_0 *such that* $PY = 0$ *and* $\dim Pc_0 = \infty$.

Proof. Part a is due to Sobczyk [18] while part b is due Pełczyński [9] (this paper of Pełczyński also contains a nice proof of Sobczyk's result). We prove now part c. Since l_1 does not have a reflexive infinite dimensional subspace the quotient space c_0/Y is not reflexive. Hence the quotient map $\phi: c_0 \to c_0/Y$ is not w compact. By a result of Pełczyński [10, Theorem 1] there is a subspace U of c_0 which is isomorphic to c_0 and such that $\phi_{|U}$ is an isomorphism. By part a there is a projection Q from c_0/Y onto ϕU. The projection $P = (\phi_{|U})^{-1}Q\phi$ has the desired properties.

Proof of Theorem 1. Let $T: Y \to c_0$ be an isomorphism into with $\dim c_0/Y = \dim c_0/TY = \infty$. By Lemma 1c there are projections P and Q in c_0 such that Pc_0 and Qc_0 are infinite dimensional and $PY = QTY = \{0\}$. Let I denote the identity operator of c_0. The space $(I - Q)c_0$ contains TY and, by Lemma 1b, it is isomorphic to c_0. Hence, by Lemma 1a, there is an operator $R: c_0 \to (I - Q)c_0$ such that $R_{|Y} = T$. Similarly there is an operator $S: c_0 \to (I - P)c_0$ auch that $S_{|TY} = T^{-1}$. Let τ be an isomorphism from c_0 onto Qc_0 (it exists by Lemma 1b). Let the operator $T_1: c_0 \to c_0$ be defined by $T_1 = R + \tau(I - SR)$. Since $(I - SR)_{|Y} = 0$. it follows that $T_{1|Y} = T$. We show next that T_1 is an isomorphism into c_0. Let $x \in c_0$, then

$$
\begin{aligned}
\| T_1 x \| &\geqq \max\left(\frac{\| QT_1 x\|}{\|Q\|}, \frac{\|(I-Q)T_1 x\|}{1+\|Q\|}\right) \\
&= \max\left(\frac{\|\tau(I-SR)x\|}{\|Q\|}, \frac{\|Rx\|}{1+\|Q\|}\right) \\
&\geqq \max\left(\frac{\|(I-SR)x\|}{\|\tau^{-1}\|\,\|Q\|}, \frac{\|Rx\|}{1+\|Q\|}\right) \\
&\geqq \max\left(\frac{\|x\| - \|S\|\,\|Rx\|}{\|\tau^{-1}\|\,\|Q\|}, \frac{\|Rx\|}{2\|Q\|}\right) \\
&\geqq \frac{\|x\|}{\|Q\|(2\|S\| + \|\tau^{-1}\|)},
\end{aligned}
$$

and this proves our assertion concerning T_1. Since T_1 is an isomorphism and $(I-P)c_0$ is a subspace of c_0 of infinite codimension it follows that $T_1(I-P)c_0$ if also a subspace of c_0 of infinite codimension which is isomorphic to c_0 (if it is infinite dimensional). By Lemma 1b there is a closed subspace U of c_0 such that U is isomorphic to c_0 and $c_0 = U \oplus T_1(I-P)c_0$. Let σ be an isomorphism from Pc_0 onto U. Define now $\tilde{T}: c_0 \to c_0$ by $\tilde{T} = \sigma P + T_1(I-P)$. It is easy to check that $\tilde{T}$ has all the required properties.

COROLLARY. *Let Y and Z be two isomorphic closed subspaces of c_0 of infinite codimension. Then there is an automorphism of c_0 which maps Y onto Z. In particular c_0/Y is isomorphic to c_0/Z.*

REMARK. Since a quantitative version of Lemma 1 is true (e.g. in part a T_0 can be chosen so that $\| T_0 \| \leqq 2 \| T \|$ and in part c) P can be chosen so that $\| P \| \leqq 3$) and all the steps of the proof given here are of a cobstructive nature (modulo the Lemma) it follows that there is a suitable $\tilde{T}$ so that $\| \tilde{T} \| \, \| \tilde{T}^{-1} \| \leqq g(\| T \| \, \| T^{-1} \|)$ where g is a certain function from $[1, \infty)$ into itself. (Actually it is not hard to obtain, by following our proof, an explicit formula for a suitable g. We do not think however that such an explicit formula will be of much interest.) The same remark applies also to Theorems 2 and 3 below.

Our next result concerns the space l_1.

THEOREM 2. *Let Y and Z be infinite dimensional closed subspaces of l_1 and let ϕ and ψ be the canonical quotient maps $\phi: l_1 \to l_1/Y$, $\psi: l_1 \to l_1/Z$. Then for every isomorphism T from l_1/Y onto l_1/Z there is an automorphism $\tilde{T}$ of l_1 such that $\psi\tilde{T} = T\phi$.*

The following diagram explains the situation:

$$\begin{array}{ccc} l_1 & \xrightarrow{\ \phi\ } & l_1/Y \\ {\scriptstyle \tilde{T}}\Big| & & \Big\downarrow{\scriptstyle T} \\ l_1 & \xrightarrow{\ \psi\ } & l_1/Z \end{array}$$

The theorem holds vacuously if l_1/Y and l_1/Z are not isomorphic. For the proof of the theorem we need the following known facts concerning l_1.

LEMMA 2. a) *Let Z and Y be Banach spaces and let ϕ be an operator from Z onto Y. Then for every operator $T: l_1 \to Y$ there exists an operator $\tilde{T}: l_1 \to Z$ such that $\phi\tilde{T} = T$ ($\tilde{T}$ is said to be a lifting of T).*

b) *Every complemented infinite dimensional subspace of* l_1 *is isomorphic to* l_1.

c) *Every infinite dimensional closed subspace of* l_1 *contains an infinite dimensional subspace complemented in* l_1.

Part a is a simple and well known consequence of the open mapping theorem (cf. Köthe [6, p. 184]). Part b is due to Pełczyński. [9]. Part c is a refinement of Pełczyński of a result which goes back to Banach (cf. [9]).

Proof of Theorem 2. Let Y, Z, T, ϕ and ψ be as in the statement of the theorem. By Lemma 2b and c there are subspaces U and V of l_1 both of which are isomorphic to l_1 such that $l_1 = U \oplus V$ and $U \subset Y$. Denote by I the identity map of l_1 and by P the projection of l_1 onto V which maps U to 0. Clearly $\phi = \phi P$ and $\phi_{|V}$ maps V onto l_1/Y. By Lemma 2a there is an operator $R: V \to l_1$ such that $\psi R = T\phi_{|V}$ and an operator $S: l_1 \to V$ such that $\phi S = T^{-1}\psi$. Let τ be an isomorphism from U onto l_1 and let $T_1: l_1 \to l_1$ be defined by

$$T_1 = (I - RS)\tau(I - P) + RP.$$

Then

$$\begin{aligned}\psi T_1 &= (\psi - \psi RS)\tau(I - P) + \psi RP \\ &= (\psi - T\phi S)\tau(I - P) + T\phi P \\ &= (\psi - TT^{-1}\psi)\tau(I - P) + T\phi \\ &= T\phi.\end{aligned}$$

The operator T_1 is surjective. Indeed,

$$T_1^* = P^*R^* + (I - P^*)\tau^*(I - S^*R^*).$$

Since $\| P^*R^*u^* \| \geqq \| R^*u^* \|$ for every $u^* \in l_1^*$, the same computation as the one we did in the proof of Theorem 1 shows that T_1 is an isomorphism into and thus T_1 is surjective.

Let Q be a projection in l_1 such that $Ql_1 \subset Z$ and $\dim Ql_1 = \dim l_1/Ql_1 = \infty$. Since T_1 is surjective it follows from Lemma 2a that Kernel T_1 is a complemented subspace of l_1, and thus l_1 can be decomposed into the direct sum (Kernel T_1) $\oplus W_1 \oplus W_2$ so that $T_1 W_1 = Ql_1$ and $T_1 W_2 = (I - Q)l_1$. Since $\psi T_1 = T\phi$ it follows that (Kernel $T_1) \oplus W_1 \subset Y$. Let σ be an isomorphism from (Kernel T_1) $\oplus W_1$ onto Ql_1 (such a σ exists since by Lemma 2b both spaces are isomorphic to l_1). Let $\tilde{T}: l_1 \to l_1$ be the operator uniquely defined by the relations

$$\tilde{T}_{|W_2} = T_{1|W_2}, \quad \tilde{T}_{|(\mathit{Kernel}\ T_1)+W_1} = \sigma.$$

It is easy to check that $\tilde{T}$ has all the required properties.

COROLLARY. *Let Y and Z be infinite-dimensional closed subspaces of l_1 such that l_1/Y is isomorphic to l_1/Z. Then Y is isomorphic to Z.*

REMARK. The proof we gave for Theorem 2 is just the dual of the argument used in the proof of Theorem 1. However, since we are dealing with non-reflexive spaces, Theorem 2 does not follow from Theorem 1 by a direct duality argument. The dual statement of Theorem 1 is as follows: If Y and Z are w^* closed subspaces of l_1 of infinite dimension then for every w^* isomorphism from l_1/Y onto l_1/Z there is a w^* automorphism $\tilde{T}$ of l_1 such that $\psi\tilde{T} = T\phi$.

We pass now to the space m. Let us first recall that an operator $T: X \to X$ is called a Fredholm operator if $\dim(\text{Kernel } T) < \infty$ and $\dim X/TX < \infty$ (as is well known these assumptions already imply that TX is closed in X). The integer $\dim(\text{Kernel } T) - \dim X/TX$ is called the index of the Fredholm operator T.

THEOREM 3. *Let Y be a closed subspace of m and let T be an isomorphism from Y into m such that $\dim m/Y = \dim m/TY = \infty$. Then*

(i) *If m/Y and m/TY are both non-reflexive, there is an automorphism $\tilde{T}$ of m which extends T* (i.e. $\tilde{T}_{|Y} = T$).

(ii) *If exactly one of the spaces m/Y and m/TY is reflexive, T cannot be extended to an automorphism of m.*

(iii) *If m/Y and m/TY are reflexive, then every extension of T to an operator from m into itself is a Fredholm operator. The index of this Fredholm operator does not depend on the particular choice of the extension and thus defines an integer-valued invariant of T which we denote by $i(T)$. The operator T has an extension to an automorphism of m if and only if $i(T) = 0$.*

Proof. The proof of part (i) of the theorem is very similar to the proof of Theorem 1. The only difference is that the special properties of c_0 exhibited in Lemma 1 have now to be replaced by the following properties of m:

LEMMA 3. a) *For every Banach spaces $Z \supset Y$ and every operator $T: Y \to m$ there is an operator from Z to m which extends T.*

b) *An infinite-dimensional subspace of m is complemented in m if and only if it is isomorphic to m.*

c) *Let Y be a closed subspace of m such that m/Y is non-reflexive. Then there is a projection P in m such that $PY = 0$ and $\dim Pm = \infty$.*

Part a of the lemma is a well-known immediate corollary of the Hahn-Banach theorem (an operator into m can be considered as a sequence of uniformly bounded linear functionals cf. [3, p. 94]). Part b was first proved in [8] (cf. also [13]). In the proof of Theorem 2 of [13] it is shown that if Y is a subspace of m such that m/Y is non-reflexive then there is a subspace U of m which is isometric to m and such that $\phi_{|U}$ is an isomorphism (where $\phi: m \to m/Y$ is the quotiemt map). From this fact part c of Lemma 3 follows immediately (cf. the proof of Lemma 1c).

The proof for part (i) of Theorem 3 is now obtained by replacing everywhere "c_0" by "m" in the proof of Theorem 1, and of course using Lemma 3 in place of Lemma 1. Part (ii) is trivial since the existence of $\tilde{T}$ implies in particular that m/Y is isomorphic to m/TY. For the proof of part (iii) we need the known

LEMMA 4. *Let $K: m \to m$ be a weakly compact operator and let $I: m \to m$ be the identity operator. Then $I + K$ is a Fredholm operator of index* 0.

Lemma 4 is a consequence of the fact that m has the so called Dunford-Pettis property and thus K^2 is a compact operator. It is well known that the classical Riesz theory for compact operators applies also to operators which have a compact power. This is the assertion of Lemma 4 (for all these consult Dunford and Schwartz [4] in particular section VII 4).

Proof of Theroem 3 part (iii). Let R and S be operators from m into itself such that $R_{|Y} = T$ and $S_{|TY} = T^{-1}$ (R and S exist by Lemma 3a). The operator $I - SR$ vanishes on Y and thus can be represented as $Q\phi$ where $\phi: m \to m/Y$ is the quotient map and Q is some operator from m/Y to m. Since m/Y is reflexive it follows that $I - SR$ is w compact and hence by Lemma 4 SR is a Fredholm operator of index 0. Similarly, since m/TY is reflexive, it follows that RS is a Fredholm operator of index 0. Thus R and S are both Fredholm operators; hence by adding suitable operators of finite-dimensional range, if necessary, we may assume without loss of generality that R and S are both maximal (i.e., are either 1-1 or onto). It now follows that index R + index $S = 0$. This proves that index R does not depend on the choice of R and thus $i(T)$ is well defived. If $i(T) = 0$, i.e., index $R = 0$, then our assumption that R is maximal implies that $\tilde{T} = R$ is an automorphism of m which extends T. Conversely, if a suitable $\tilde{T}$ exists then index $\tilde{T} = i(T) = 0$. This concludes the proof of Theorem 3.

REMARKS. 1. Case (ii) may actually happen. It is well known for example that there is a subspace Y of m such that m/Y is isomorphic to l_2. Since $m \oplus m$ is isomorphic to m it follows that there is a subspace Z of m which is isomorphic to Y such that $m/Z \approx m \oplus l_2$. The same reasoning shows that whenever $Y \subset m$ is such that m/Y is reflexive there exists an isomorphism T from Y into m which cannot be extended to an automorphism of m. The first part of the proof of Theorem 1 shows, however, that if m/Y is reflexive and T is an isomorphism of Y into m with m/T non-reflexive then there is an isomorphism of m into m which extends T. Thus in particular $m/TY \approx m \oplus m/Y$. (In fact, if Y and Z are isomorphic subspaces of m with m/Y reflexive, then either $m/Z \approx m \oplus m/Y$ or there exists a finite dimensional space F such that $m/Z \approx F \oplus m/Y$ or $m/Y \approx F \oplus m/Z$. See the next remark.)

2. Every integer can arise as $i(T)$ for a suitable T for which case (iii) holds. Indeed let $S: m \to m$ be the shift operator defined by $S(\lambda_1, \lambda_2, \cdots) = (0, \lambda_1, \lambda_2, \cdots)$, and let Y be a subspace of m such that m/Y is infinite dimensional and reflexive. Then for every positive integer k $S^k_{|Y}$ is an isomorphism for which case (iii) holds and $i(S^k_{|Y}) = -k$. To obtain a T with $i(T)$ positive we just have to consider $S^{-k}_{|Y}$ where Y is a subspace of $S^k m$ such that $S^k m/Y$ is infinite dimensional and reflexive. If in particular $Y \subset m$ is chosen so that $m/Y \approx l_2$ then also $m/S^k Y = l_2$ $(k = 1, 2, \cdots)$ and thus in case (iii) the additional assumption $m/Y \approx m/TY$ does not ensure the existence of an automorphism $\tilde{T}$ which extends T. Actually it seems to be an open problem whether in case (iii) it can happen that $m/Y \not\approx m/TY$. It can be proved that this question is equivalent to the following special case of a well-known problem of Banach: Let Z be an infinite-dimensional reflexive subspace of $L_1(0,1)$. Is Z always isomorphic to $Z \oplus R$ where R is the one dimensional space?

3. The validity of statement (i) of Theorem 3 in the special case where $Y = c_0$ and $TY \subset c_0$ was conjectured by Berg [2]. Berg proved this conjecture in [2] under the additional assumption that $\dim c_0/Tc_0 < \infty$. The conjecture of Berg can be proved by a simpler argument than the one we gave for the general case. Indeed, in this special case T^{**} is an isomorphism of m into m whose restriction to c_0 is T. Hence the first part of the proof given here (the construction of T_1) is not needed in the proof of Berg's conjecture.

In view of Theorem 3 it is desirable to have a criterion for deciding whether a given subspace of m is of reflexive codimension. A criterion of this type is given in our next result.

THEOREM 4. *Let Y be a subspace of m such that m/Y is reflexive. Then Y has a subspace isomorphic to m.*

Proof. We note first that Y contains a subspace U with $U \approx c_0$. Indeed, if no such U exists, then Y and c_0 contain no isomorphic infinite dimensional subspaces and thus by [14] $Y + c_0$ is a closed subspace of m with $\dim Y \cap c_0 < \infty$. But then c_0 would be isomorphic to a subspace of m/Y, a contradiction. Further, by Theorem 3(i) we may assume without loss of generality that Y contains c_0 itself. Indeed, there is an automorphism τ of m such that $\tau U = c_0$ and if τY can be shown to contain a subspace isomorphic to m the same will be true for Y.

We shall prove that there is an infinite subset E of the integers N such that $m(E)$ (= all elements of m which vanish outside E) is contained in Y. Since $m(E) \approx m$ this will conclude the proof.

Let $\{E_\alpha\}_{\alpha \in A}$ be an uncountable collection of infinite subsets of N such that $E_\alpha \cap E_\beta$ is finite for every $\alpha \neq \beta$. Assume now that for every α there is a $y_\alpha \in m(E_\alpha)$ such that $\| y_\alpha \| = 1$ and $y_\alpha \notin Y$. Let $\phi: m \to m/Y$ be the quotient map. Since A is uncountable there is a $\delta > 0$ and an infinite subset $A' \subset A$ such that $\| \phi(y_\alpha) \| \geqq \delta$ for $\alpha \in A'$. Choose any sequence $\{\alpha_i\}_{i=1}^{\infty}$ in A'. Since $c_0 \subset Y$ it follows that for every scalars $\{a_i\}_{i=1}^{n}$, $\| \sum_{i=1}^{n} a_i \phi(y_\alpha) \| \leqq \max_i |a_i|$ and thus $T: c_0 \to m/Y$ defined by $T(\lambda_1, \lambda_2, \cdots) = \sum_{i=1}^{\infty} \lambda_i \phi(y_{\alpha_i})$ is a bounded operator. Since m/Y is reflexive T is weakly compact and thus compact. (Recall the well known facts (cf. e.g. [3]) that in l_1 a sequence converges weakly if and only if it converges strongly and that T is compact [resp. w compact] if and only if T^* has the same property). It follows (by passing to a subsequence if necessary) that $\phi(y_{\alpha_i})$ converges in norm to an element $z \in m/Y$. This however contradicts the facts that $\| \phi(y_{\alpha_i}) \| \geqq \delta$ for every i and $\| \sum_{i=1}^{n} \phi(y_{\alpha_i}) \| \leqq 1$ for every n. Q.E.D.

COROLLARY. *Let Y be a subspace of m such that no subspace of Y is isomorphic to m. (In particular Y may be any separable subspace of m.) Then every isomorphism of Y into m has an extension to an automorphism of m.*

Proof. Use Theorem 3(i) and Theorem 4.

REMARK. Theorem 4 shows that a subspace Y of m such that m/Y is reflexive must be a "large" subspace of m. However, m/Y may also be quite "large". In [15] it is proved that if Γ has the cardinality of the continuum then $l_2(\Gamma)$ is isomorphic to a quotient space of m.

3. Remarks and open problems.

a. **Subspace homogeneous Banach spaces.** We call a Banach space X subspace homogeneous if for every two isomorphic subspaces Y and Z of X, both of infinite codimension, there is an automorphism of X which carries Y onto Z. As mentioned already in the introduction we conjecture that c_0 and l_2 are the only subspace homogeneous Banrch spaces. At present we cannot even prove that this property characterizes c_0 and l_2 among the classical Banach spaces. Here are some partial results in this direction. All these results are simple consequences of known results. The space $C(K)$ of continuous functions on a compact metric space K is not subspace homogeneous unless it is isomorphic to c_0. Indeed, it follows from a result of Amir [1] and Pełczyński [11, section 9] that if $C(K) \not\approx c_0$ then there are subspaces Y and Z of $C(K)$ with Y complemented, Z uncomplemented and $Y \approx Z \approx C(\omega^\omega)$ (= the space of continuous functions on the set of ordinals $\{\xi; 0 \leqq \xi \leqq \omega^\omega\}$ in the order topology). (Moreover, Y and Z may be chosen such that $C(K)/Y \approx C(K)/Z$; this bears on the remarks in b. below).

By a result of Rudin [17] and Rosenthal [12, p. 52] $L_p(0,1)$ has for $1 < p < \frac{4}{3}$ an uncomplemented subspace isomorphic to l_2. Since, as is well known, $L_p(0,1)$ also has complemented subspaces isomorphic to l_2 (see e.g. [12] and its references) it follows that $L_p(0,1)$ is not subspace homogeneous for $1 < p < \frac{4}{3}$. The same result of Rudin and Rosenthal combined with [9, Proposition 7] and a simple compactness argument show that for $1 < p < \frac{4}{3}$, l_p has an uncomplemented subspace isomorphic to l_p, and thus for these values of p, l_p is not subspace homogeneous. It is likely that the result of Rudin and Rosenthal can be extended to every p in the open interval (1.2) but at present this remains an open question.

By a result of Kadec [5], $L_p(0,1)$ has a subspace $X = X(p,r)$ such that $L_p(0,1)/X \approx l_r$ whenever $2 \leqq r \leqq p$. Since $L_p(0,1) \oplus L_p(0,1) \approx L_p(0,1)$ it follows that if $p > r > 2$ then $L_p(0,1)$ has isomorphic subspaces Y and Z such that $L_p(0,1)/Z \approx l_r$ and $L_p(0,1)/Y \approx l_r \oplus L_p(0,1)$. Since $l_r \not\approx l_r \oplus L_p(0,1)$ it follows that $L_p(0,1)$ is not subspace homogeneous for $2 < p < \infty$. We cannot prove a similar statement for l_p, $2 < p < \infty$.

Since every separable Banach space is a quotient space of l_1 and $l_1 \oplus l_1 \approx l_1$ it follows that there are isomorphic subspaces Y and Z in l_1 such that $l_1/Y \approx c_0$ and $l_1/Z \approx c_0 \oplus l_1$. Since $c_0 \not\approx c_0 \oplus l_1$ we get that l_1 is not subspace homogeneous. A similar argument shows that $L_1(0,1)$ is not subspace homogeneous.

We considered above only separable spaces X. However, we cannot prove that a subspace homogeneous Banach space is necessarily separable.

b. **Quotient homogeneous spaces.** We call a Banach space X quotient homogeneous if whenever Y and Z are two infinite dimensional subspaces of X with $X/Y \approx X/Z$ then there is an automorphism of X which carries Y onto Z. We conjecture that l_1 and l_2 are the only quotient homogeneous Banach spaces. For reflexive X it is clear that X is subspace homogeneous if and only if X^* is quotient homogeneous. Hence for reflexive X this conjecture is the same as the one discussed in the preceding paragraphs. The results of Amir and Pełczyński mentioned above show also that a $C(K)$ space is not quotient homogeneous if $C(K) \not\approx c_0$ (K compact metric). We cannot however prove our conjecture for $X = c_0$.

The space $L_1(0,1)$ is not quotient homogeneous. Indeed, let U be a subspace of l_1 such that $l_1/U \approx L_1(0,1)$. Since $l_1 \oplus L_1(0,1) \approx L_1(0,1)$ and $U \oplus L_1(0,1) \not\approx L_1(0,1)$ (by [7]) it follows that $L_1(0,1)$ has a subspace Y with $L_1(0,1)/Y \approx L_1(0,1)$ and $Y \approx U \oplus L_1(0,1) \not\approx L_1(0,1)$. Since clearly there is also a Z in $L_1(0,1)$ with $Z \approx L_1(0,1)$ and $L_1(0,1)/Z \approx L_1(0,1)$ the desired result follows.

c. **Extension of projections.** The results of Section 2 also show the following: Let Y be a closed subspace of c_0 and let $P\colon Y \to Y$ be a projection in Y. Then there is a projection $\tilde{P}$ in c_0 such that $\tilde{P}_{|Y} = P$. If $\dim c_0/Y < \infty$ the existence of $\tilde{P}$ is obvious; take e.g. $\tilde{P} = PQ$ where Q is a projection of c_0 onto Y. If $\dim c_0/Y = \infty$ we proceed as follows: Let U and V be the subspaces of $c_0 \oplus c_0$ defined by $U = \{(x,0); x \in PY\}$, $V = \{(0,x); x \in (I-P)Y\}$. ($I$ denotes the identity map of Y). For $j = 1,2$, let $i_j\colon c_0 \oplus c_0 \to c_0$ be defined by $i_j(x_1,x_2) = x_j$. By Theorem 1 there is a surjective isomorphism $\tau\colon c_0 \oplus c_0 \to c_0$ such that $\tau | U = i_1 | U$ and $\tau | V = i_2 | V$. Let Q be the projection on $c_0 \oplus c_0$ defined by $Q(x_1,x_2) = (x_1,0)$. The operator $\tilde{P} = \tau Q \tau^{-1}$ has all the desired properties. The same argument shows that if $Y \subset m$ with m/Y non-reflexive then every projection in Y can be extended to a projection in m. We do not know whether this holds also if m/Y is reflexive. The question involved here is the following: Let $Y \subset m$ with m/Y reflexive and let Z be a complemented infinite dimensional subspace of Y. Does there exist a subspace U of m such that $U \approx Z$ with m/U reflexive?

For l_1 we get the following result: Let Y be a closed subspace in l_1 and let $\phi\colon l_1 \to l_1/Y$ be the quotient map. Let P be a projection in l_1/Y. Then there is a projection $\tilde{P}$ in l_1 such that $\phi\tilde{P} = P\phi$.

d. **The spaces $c_0(\Gamma)$, $l_1(\Gamma)$ and $m(\Gamma)$.** Some of the results proved in Section 2 can be extended to the natural generalizations of c_0, l_1 and m namely to the spaces $c_0(\Gamma)$, $l_1(\Gamma)$ and $m(\Gamma)$ for uncountable Γ. For example by using a generalization of Lemma 2 (cf. [6] and also [16]) the argument in the proof of Theorem 2 shows that if Y and Z are subspaces of $l_1(\Gamma)$ with $\dim Y = \dim Z = \operatorname{card}\Gamma$ then for every surjective isomorphism $T: l_1(\Gamma)/Y \to l_1(\Gamma)/Z$ there is an automorphism $\tilde{T}$ of $l_1(\Gamma)$ such that $\psi\tilde{T} = T\phi$ where $\phi: l_1(\Gamma) \to l_1(\Gamma)/Y$ and $\psi: l_1(\Gamma) \to l_1(\Gamma)/Z$ are the quotient maps. For $m(\Gamma)$ the arguments in Section 2 together with a generalization of Lemma 3 (cf. [16]) show the following: Let $Y \subset m(\Gamma)$ and let T be an isomorphism of Y into $m(\Gamma)$. Assume that there exists a subspace U of $m(\Gamma)$ such that $U \approx c_0(\Gamma)$ and $U \perp TY$ (i.e., $U \cap TY = \{0\}$ and $U + TY$ closed in $m(\Gamma)$). Then T can be extended to an isomorphism of $m(\Gamma)$ into itself. If, in addition, there is a $V \subset m(\Gamma)$ with $V \approx c_0(\Gamma)$ and $V \perp Y$, then T has an extension to an automorphism of $m(\Gamma)$. Such U and V always exist if $\dim Y < \dim m(\Gamma) = 2^{card(\Gamma)}$. For $c_0(\Gamma)$ the situation is less clear since we do not know whether every operator from a subspace of $c_0(\Gamma)$ into $c_0(\Gamma)$ can be extended to an operator of $c_0(\Gamma)$ into $c_0(\Gamma)$.

e. **Injective Banach spaces.** A Banach space X is said to be injective if it satisfies the conclusion of Lemma 3a (where "m" is replaced by "X"). By using the results of [13], the first part of the argument of Theorem 1 may be used to show the following: Let X be infinite dimensional and injective, let A be a closed subspace of m, and let $T: A \to X$ be an isomorphism into. Then if $X/T(A)$ is not reflexive, there exists an into isomorphism $\tilde{T}: m \to X$, extending T. (Moreover if $X/T(A)$ is reflexive, then it can be proved that X is isomorphic to m, and thus Theorem 3 applies directly in this case).

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